

# Spin Non-commutativity and the Three-Dimensional Harmonic Oscillator

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A three-dimensional harmonic oscillator with spin non-commutativity in the phase space is considered. The system has a regular symplectic structure and by using supersymmetric quantum mechanics techniques, the ground state is calculated exactly. We find that this state is infinitely degenerate and it has explicit spontaneous broken symmetry. Analyzing the Heisenberg equations, we show that the total angular momentum is conserved.

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In the last ten years the implications of non-commutative geometry [1] has been intensively investigated as a way, for example, to develop computational techniques for understanding problems that go beyond perturbation theory. A particularly interesting example is the quantum Hall effect where – due to the fact that the magnetic field is strong – non-perturbative techniques are required. Besides, the study of motion of charged particles in strong magnetic fields [2, 3] is an interesting mathematical problem because, as it is known, the commutator of the momenta (more precisely the covariant derivatives), is different from zero. In other words, this is a genuine example of a non-commutative geometry problem.

Some years ago, Nair and Polychronakos [4] studied the noncommutative harmonic oscillator, *i.e.* a system described by the Hamiltonian

$$\hat{H} = \frac{1}{2} (\mathbf{p}^2 + \mathbf{r}^2), \quad (1)$$

and the deformed commutators [5] given by

$$\begin{aligned} [\hat{x}_i, \hat{x}_j] &= i\theta_{ij}, \\ [\hat{p}_i, \hat{p}_j] &= iB_{ij}, \\ [x_i, p_j] &= i\delta_{ij}. \end{aligned} \quad (2)$$

with  $\{i, j\} \in \{1, 2, 3\}$ . This problem is exactly soluble and presents a fixed point.

In three dimensions, for example, if we choose  $\theta_{ij} = \epsilon_{ijk}\theta_k$  and  $B_{ij} = \epsilon_{ijk}B_k$ , with  $\theta_i, B_j$  constant vectors, then the determinant of the symplectic matrix  $\Omega_{ab} = [\xi_a, \xi_b]$  – with  $\{\xi_a\} \equiv (\mathbf{x}, \mathbf{p})$  – turn out to be  $\det(\Omega) = -(1 - \boldsymbol{\theta} \cdot \mathbf{B})^2$  which, therefore, vanishes for  $\boldsymbol{\theta} \cdot \mathbf{B} = 1$  [4]. Alternatively, it is possible to perform a coordinate transformation in the phase space in order to get a well behaved symplectic matrix, but in such a case, the coordinate transformation turn out to be singular in the parameter space  $\{\theta_i, B_k\}$ , again for  $\boldsymbol{\theta} \cdot \mathbf{B} = 1$ .

An interesting problem, occurs when the spatial variables are mixed with the spin variables in such a way that the antisymmetric tensors in the right hand sides of Eqs. (2) are given by

$$\theta_{ij} = \theta^2 \epsilon_{ijk} \hat{s}_k, \quad B_{ij} = \kappa^2 \epsilon_{ijk} \hat{s}_k, \quad (3)$$

where  $\theta$  and  $\kappa$  are length and momentum scales, respectively (in natural units with  $\hbar = 1$ ), and  $\hat{s}_k$  are spin matrices.

The deformation of the second commutator in (2) with  $B_{ij}$  specified in (3) is a kind of non-relativistic [6] version of the Snyder-Yang algebra [7, 8]. Indeed, the choice (3) corresponds to the reduction of  $M_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$  to just  $\sim i\epsilon_{ijk}\sigma_k$ .

The complete set of consistent commutation relations

$$\begin{aligned} [\hat{x}_i, \hat{x}_j] &= i\theta^2 \epsilon_{ijk} \hat{s}_k & [\hat{p}_i, \hat{p}_j] &= i\kappa^2 \epsilon_{ijk} \hat{s}_k \\ [\hat{x}_i, \hat{p}_j] &= i(\delta_{ij} + \kappa\theta \epsilon_{ijk} \hat{s}_k) & [\hat{s}_i, \hat{s}_j] &= i\epsilon_{ijk} \hat{s}_k \\ [\hat{x}_i, \hat{s}_j] &= i\theta \epsilon_{ijk} \hat{s}_k & [\hat{p}_i, \hat{s}_j] &= i\kappa \epsilon_{ijk} \hat{s}_k \end{aligned} \quad (4)$$

can be explicitly realized in terms of canonical variables through the shift

$$\begin{aligned} \hat{x}_i &\rightarrow \hat{x}_i = x_i + \theta s_i, \\ \hat{p}_i &\rightarrow \hat{p}_i = p_i + \kappa s_i, \\ \hat{s}_i &\rightarrow \hat{s}_i = s_i = \frac{1}{2}\sigma_i, \end{aligned} \quad (5)$$

where  $(x_i, p_j)$  obey the standard Heisenberg algebra and the identification  $s_i = \frac{1}{2}\sigma_i$  corresponds to spin-1/2, the situation we shall consider in this paper.

These noncommutative variables give rise to the symplectic matrix

$$\begin{pmatrix} [\hat{x}_i, \hat{x}_j] & [\hat{x}_i, \hat{p}_j] & [\hat{x}_i, \hat{s}_j] \\ [\hat{p}_i, \hat{x}_j] & [\hat{p}_i, \hat{p}_j] & [\hat{p}_i, \hat{s}_j] \\ [\hat{s}_i, \hat{x}_j] & [\hat{s}_i, \hat{p}_j] & [\hat{s}_i, \hat{s}_j] \end{pmatrix} \quad (6)$$

which is regular, being its determinant a nonvanishing constant (independent of  $\theta$  and  $\kappa$ )<sup>1</sup>. This is a very interesting peculiarity of the oscillator with non-commutativity of spin, which justifies a more careful analysis of their properties.

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<sup>1</sup> The same is also true for, at least, the model with spin 1.

Thus, following the prescription in Eq. (5), the Schrödinger equation is

$$\hat{H}(x_i + \theta \frac{\sigma_i}{2}, p_i + \kappa \frac{\sigma_i}{2})|\psi(t)\rangle = i\partial_t|\psi(t)\rangle. \quad (7)$$

The model based on the harmonic oscillator we are considering in the present paper is then defined by the Hamiltonian [9]

$$\begin{aligned} \hat{H} &= \frac{1}{2} (\hat{\mathbf{p}}^2 + \hat{\mathbf{r}}^2) \\ &= \frac{1}{2} (\hat{p}_i + i\hat{x}_i)(\hat{p}_i - i\hat{x}_i) + \frac{3}{2} \\ &= A_i^\dagger A_i + E_0, \end{aligned} \quad (8)$$

where  $E_0$  is the energy of the ground state and

$$\begin{aligned} A_i &= \frac{1}{\sqrt{2}} (\hat{p}_i - i\hat{x}_i), \\ A_i^\dagger &= \frac{1}{\sqrt{2}} (\hat{p}_i + i\hat{x}_i) \end{aligned} \quad (9)$$

are the analogous of the creation and destruction operators of the usual commutative case, but now they satisfy the algebra

$$\begin{aligned} [A_i, A_j] &= \frac{i}{2} (\kappa - i\theta)^2 \epsilon_{ijk} s_k, \\ [A_i^\dagger, A_j^\dagger] &= \frac{i}{2} (\kappa + i\theta)^2 \epsilon_{ijk} s_k, \\ [A_i, A_j^\dagger] &= \delta_{ij} + i(\kappa^2 + \theta^2) \epsilon_{ijk} s_k, \end{aligned} \quad (10)$$

which is smooth in the commutative limit, *i.e.* when  $\theta, \kappa \rightarrow 0$ .

To further discuss the physical content of this generalization of the harmonic oscillator with such a nonconventional algebraic structure, it is useful to consider the supersymmetric version associated to (8) and study, for example, the ground state of the model.

In principle this is a direct calculation: firstly we redefine the zero energy level by subtracting the constant  $E_0$  to the Hamiltonian, *i.e.* we change  $\hat{H} \rightarrow \hat{H} + E_0$  so that (9) changes into

$$\hat{H} = A_i^\dagger A_i, \quad (11)$$

which is a positive semi-definite operator.

Following the supersymmetrization procedure proposed in [10, 11] (see also [12]), the supercharges are defined as follows

$$\begin{aligned} A_i &\rightarrow Q = A_i \psi_i = A_i \sigma_i \otimes \sigma_- \equiv A \otimes \sigma_-, \\ A_i^\dagger &\rightarrow Q^\dagger = A_i^\dagger \psi_i^\dagger = A_i^\dagger \sigma_i \otimes \sigma_+ \equiv A \otimes \sigma_+, \end{aligned} \quad (12)$$

where  $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$  fulfil  $\sigma_\pm^2 = 0$  and, therefore,  $\psi_i^2 = 0 = \psi_i^{\dagger 2}$ .

The supersymmetric Hamiltonian is then defined as

$$H = \frac{1}{2} \{Q^\dagger, Q\} = \frac{1}{2} A^\dagger A \otimes \frac{\mathbf{1}_{2 \times 2} + \sigma_3}{2} + \frac{1}{2} A A^\dagger \otimes \frac{\mathbf{1}_{2 \times 2} - \sigma_3}{2}, \quad (13)$$

$$= \frac{1}{2} \left( \mathbf{p}^2 + \mathbf{r}^2 + 3\theta \boldsymbol{\sigma} \cdot \mathbf{r} + 3\kappa \boldsymbol{\sigma} \cdot \mathbf{p} + \frac{9}{4}(\theta^2 + \kappa^2) \right) \otimes \mathbf{1}_{2 \times 2} - \left( \frac{3}{2} + \boldsymbol{\sigma} \cdot \mathbf{L} \right) \otimes \sigma_3, \quad (14)$$

which commutes with the supercharges

$$[H, Q] = 0 = [Q^\dagger, H].$$

This is the standard supersymmetric algebra.

The last term of (14) is the spin-orbit coupling that emerges from the usual three-dimensional supersymmetry, while  $\boldsymbol{\sigma} \cdot \mathbf{r}$  and  $\boldsymbol{\sigma} \cdot \mathbf{p}$  correspond to magnetic dipolar and Dresselhaus [15] interactions respectively.

In order to find the ground states one note that the supercharge  $Q$  annihilates the vacuum, namely

$$Q \Psi_0 = 0, \quad (15)$$

where  $\Psi_0$  denotes the ground state. The previous equation becomes

$$A_i \sigma_i \otimes \sigma_- \Psi_0 = 0 \quad (16)$$

or, more explicitly,

$$\begin{pmatrix} 0 & 0 \\ A_i \sigma_i & 0 \end{pmatrix} \begin{pmatrix} \Psi_0^I \\ \Psi_0^{II} \end{pmatrix} = 0 \Rightarrow A_i \sigma_i \Psi_0^I = 0, \quad (17)$$

with  $\{\Psi_0^I, \Psi_0^{II}\}$  two-components spinors.

In matrix form, Eq. (17) writes as

$$[\boldsymbol{\sigma} \cdot (\mathbf{p} - i\mathbf{x}) + M] \Psi_0^I = 0, \quad (18)$$

where  $M = \frac{3}{2}(\kappa - i\theta)$  is a complex number.

Equation (18) is solved by functions of the form

$$\Psi_0^I = e^{-\frac{\mathbf{x}^2}{2} + i\mathbf{k} \cdot \mathbf{x}} u(\mathbf{k}), \quad (19)$$

where  $u(\mathbf{k})$  is a constant two-components spinor satisfying the conditions

$$(\boldsymbol{\sigma} \cdot \mathbf{k} + M) u(\mathbf{k}) = 0, \quad (20)$$

with  $\mathbf{k} = \mathbf{k}_R + i\mathbf{k}_I \in \mathbb{C}^3$ , a complex vector.

The condition (20) implies that  $\mathbf{k}^2 = M^2$  and then

$$\begin{aligned} \mathbf{k}_R^2 - \mathbf{k}_I^2 &= \frac{9}{4}(\kappa^2 - \theta^2), \\ \mathbf{k}_R \cdot \mathbf{k}_I &= -\frac{9}{4}\kappa\theta. \end{aligned} \quad (21)$$

These equations are the same we found in a previous work [12].

Notice that the conditions on  $\mathbf{k} = \mathbf{k}_R + i\mathbf{k}_I$  determine only the square  $\mathbf{k}^2$ . Then, there is a continuous of degenerate minima which can be obtained from one of them by a unitary transformation given by  $3 \times 3$  complex matrices satisfying  $U^t U = \mathbf{1}_3$ . Near the identity,  $U = e^{iM} \simeq \mathbf{1}_3 + iM$ , the previous condition requires  $M^t = -M$ . Therefore, the Lie algebra of this group is the space of complex and antisymmetric  $3 \times 3$  matrices. This corresponds to the Lie algebra of the complexification  $\overline{SO(3)}$  of  $SO(3)$ , which has the covering group  $SL(2, \mathbb{C})$ . This Lie algebra is generated by  $3 \times 3$  matrices  $\{X_i, \bar{X}_i, i = 1, 2, 3\}$  which satisfy the commutation relations  $[X_i, X_j] = i\epsilon_{ijk} X_k$ ,  $[X_i, \bar{X}_j] = i\epsilon_{ijk} \bar{X}_k$  and  $[\bar{X}_i, \bar{X}_j] = -i\epsilon_{ijk} X_k$ .

Thus, for a given  $\mathbf{k} \in \mathbb{C}^3$ , the solution for the ground state in Eq. (19) has the symmetry corresponding to rotations around this given direction and, therefore, its little group is  $SO(2) \times SO(2)$ , generated by  $\mathbf{k} \cdot \mathbf{X}$  and  $\mathbf{k} \cdot \bar{\mathbf{X}}$ . So, the symmetry of the Hamiltonian  $\overline{SO(3)}$  is broken down to  $SO(2) \times SO(2)$ , and the transformations which move from one ground state to another are elements of the quotient group  $\overline{SO(3)} / (SO(2) \times SO(2))$ .

Let us discuss now the problem concerning to the dynamical evolution of the operators  $\hat{x}, \hat{p}$  and  $\hat{s}$ . This evolution is determined by the Heisenberg equations for such operators, *i.e.*

$$\dot{\hat{x}}_i = \frac{1}{i}[\hat{x}_i, H], \quad \dot{\hat{p}}_i = \frac{1}{i}[\hat{p}_i, H], \quad \dot{\hat{s}}_i = \frac{1}{i}[\hat{s}_i, H], \quad (22)$$

which, for the Hamiltonian (14) under consideration, are

$$\begin{aligned} \dot{\mathbf{r}} \otimes \mathbf{1}_{2 \times 2} &= (\mathbf{p} + 3\kappa \mathbf{s}) \otimes \mathbf{1}_{2 \times 2} + 2(\mathbf{r} \times \mathbf{s}) \otimes \sigma_3, \\ \dot{\mathbf{p}} \otimes \mathbf{1}_{2 \times 2} &= -(\mathbf{r} + 3\theta \mathbf{s}) \otimes \mathbf{1}_{2 \times 2} + 2(\mathbf{p} \times \mathbf{s}) \otimes \sigma_3, \\ \dot{\mathbf{s}} \otimes \mathbf{1}_{2 \times 2} &= 3(\kappa \mathbf{p} + \theta \mathbf{r}) \times \mathbf{s} \otimes \mathbf{1}_{2 \times 2} - 3(\mathbf{L} \times \mathbf{s}) \otimes \sigma_3. \end{aligned} \quad (23)$$

These equations imply the conservation of the total angular momentum. Indeed, from the first two equations in (23) one gets

$$\dot{\mathbf{L}} \otimes \mathbf{1}_{2 \times 2} = -3(\kappa \mathbf{p} + \theta \mathbf{r}) \times \mathbf{s} \otimes \mathbf{1}_{2 \times 2} + 3(\mathbf{L} \times \mathbf{s}) \otimes \sigma_3$$

and using the third equation in (23) one finds

$$\frac{d}{dt}(\mathbf{L} + \mathbf{s}) = 0, \quad (24)$$

the aforementioned result.

On the other hand, the different terms appearing in the Heisenberg equations of this extension of the three-dimensional supersymmetric harmonic oscillator are relevant in different contexts. To see this, note first that the Hamiltonian in Eq. (14) is block-diagonal, acting each block on the corresponding two-component spinor,  $\Psi^I$  and  $\Psi^{II}$ . But, as we have seen, only the lower component has a normalizable ground state at zero energy. The equations of motion restricted to this sector reduce to

$$\begin{aligned} \dot{\mathbf{r}} &= (\mathbf{p} + 3\kappa\mathbf{s}) + 2(\mathbf{r} \times \mathbf{s}), \\ \dot{\mathbf{p}} &= -(\mathbf{r} + 3\theta\mathbf{s}) + 2(\mathbf{p} \times \mathbf{s}), \\ \dot{\mathbf{s}} &= 3(\kappa\mathbf{p} + \theta\mathbf{r}) \times \mathbf{s} - 3(\mathbf{L} \times \mathbf{s}). \end{aligned} \quad (25)$$

Here, the last three terms and on the right sides are simply corrections due to supersymmetry and they are present regardless of the non-commutativity of spin. The couplings  $\mathbf{r} \times \mathbf{s}$  and  $\mathbf{p} \times \mathbf{s}$  correspond to magnetic dipolar and Rashba forces respectively.

The terms coming from the noncommutative nature of the problem open the doors for interesting cases. Consider for example the algebra (4) with  $\kappa \neq 0$  and  $\theta = 0$ . In this case, the relevant commutator [16]

$$[\hat{p}_i, \hat{p}_j] = i\kappa^2 \epsilon_{ijk} \hat{s}_k,$$

is the analog of the Landau problem for a generator  $SU(2)$  instead of  $U(1)$ . In fact, the part of the Hamiltonian corresponding to the free particle, once the shift (5) is performed, turn out to be

$$H = \frac{1}{2m} \left( \mathbf{p} + \frac{\kappa}{2} \boldsymbol{\sigma} \right)^2,$$

which describes the motion of a particle of mass  $m$  interacting with a chromo-magnetic constant field in the sense that, in the field intensities tensor  $F_{\mu\nu}$ , with

$$F_{12} = \partial_1 A_2 - \partial_2 A_1 - i[A_1, A_2],$$

the choice  $A_1 = \kappa\sigma_1$  and  $A_2 = \kappa\sigma_2$  leads to

$$F_{12} = \kappa^2 \sigma_3. \quad (26)$$

The Hamiltonian in Eq. (26) has been recently employed to model the graphene [16].

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